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LETTER TO THE EDITOR

Exact solutions to the two-dimensional Korteweg–de Vries–Burgers equation

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Abstract. Two methods are described for obtaining exact travelling monotone shock wave solutions to the two-dimensional Korteweg–de Vries–Burgers equation. They are compared with two methods that have been reported recently by other authors.

In this article we discuss travelling monotone shock wave solutions to the two-dimensional Korteweg–de Vries–Burgers (2D-KdVB) equation

$$(u_t + uu_x + \mu u_{xxx} - \nu u_{xx})_x + su_{yy} = 0 \tag{1}$$

where μ, ν are real constants and $s = \pm 1$. Recently solutions to (1) were derived by two different methods [1,2]. The purpose of this article is to show how these solutions are related and to rederive them using more economical methods.

Before discussing the two-dimensional problem we review briefly the derivation of solutions to the 1D-KdVB equation

$$u_t + uu_x + \mu u_{xxx} - \nu u_{xx} = 0. \tag{2}$$

Jeffrey and co-workers [3,4] proposed the transformation

$$u = p(\ln F)_x + q(\ln F)_{xx} \tag{3}$$

where p and q are constants. The choice $p = -12\nu/5$ and $q = 12\mu$ leads to an equation in bilinear form for F. Travelling wave solutions to (2) are sought by taking $F = 1 + \exp[\xi - \xi_0]$, where $\xi = kx - \omega t$, ξ_0 is an arbitrary constant, and k and ω are constants to be determined. Two solutions are obtained in terms of

$$S = \operatorname{sech}[\tfrac{1}{2}(\xi - \xi_0)] \qquad \text{and} \qquad T = \tanh[\tfrac{1}{2}(\xi - \xi_0)] \tag{4}$$

namely

$$u = \frac{3\nu^2}{25\mu}[S^2 - 2T \pm 2] \qquad \text{with} \qquad \xi = \frac{\nu}{5\mu}\left(x \mp \frac{6\nu^2}{25\mu}t\right). \tag{5}$$

Halford and Vlieg-Hulstman [5] have shown that the second of these solutions may also be obtained via the transformation

$$u = -\frac{12\mu F_x^2}{F^2}. \tag{6}$$

A different (and arguably more efficient) approach is to assume from the outset that $u(x,t) = U(\xi)$; substitution into (2) and one integration gives

$$-cU + \tfrac{1}{2}U^2 + \mu k^2 U_{\xi\xi} - \nu k U_\xi = C \qquad \text{where} \qquad c = \omega/k \tag{7}$$

and C is an integration constant. Note that setting $C = 0$ in (7) is equivalent to demanding that $u \to 0$ either as $\xi \to +\infty$ or as $\xi \to -\infty$; the solutions (5) have this property. Equation (7) cannot be integrated directly. McIntosh [6] and Samsonov and Sokurinskaya [7] used an Ince transformation [8]

$$U = rz^2 + \beta \qquad z = e^\xi \qquad r = r(z) \tag{8}$$

to reduce (7) to a directly integrable differential equation for r. The resulting solutions for u are

$$u = -\frac{12\nu^2}{25\mu}\frac{1}{\{1+\exp[-(\xi-\xi_0)]\}^2} + c + \frac{6\nu^2}{25\mu} \tag{9}$$

where

$$\xi = \frac{\nu}{5\mu}(x - ct) \qquad \text{and} \qquad c = \pm\left[\left(\frac{6\nu^2}{25\mu}\right)^2 - 2C\right]^{1/2}.$$

Xiong [9] obtained (9) from (7) by choosing the fortuitous ansatz

$$U(\xi) = \frac{\alpha}{(1+\exp[-(\xi-\xi_0)])^2} + c + (c^2 + 2C)^{1/2}$$

where α, k and c are constants to be determined. The solutions (9) may be written in the alternative form

$$u = \frac{3\nu^2}{25\mu}[S^2 - 2T] + c \tag{10}$$

with ξ and c as in (9). Note that the solutions (5) are just the solutions (10) with $C = 0$. Vlieg-Hulstman and Halford [10] reviewed the results in [3, 4, 6, 9]. They observed that (2) is invariant under the transformation $u^* = u + \lambda$, $x^* = x + \lambda t$, $t^* = t$, where λ is a constant. With

$$\lambda = \mp\frac{6\nu^2}{25\mu} \pm \left[\left(\frac{6\mu^2}{25\mu}\right)^2 - 2C\right]^{1/2} \tag{11}$$

the solutions (5) are transformed into the solutions (10). As well as using the transformation (3), Jeffrey and Mohamad [4] described a direct method in which they assumed that

$$U(\xi) = AS^n + BT^m + D. \tag{12}$$

Substitution of (12) into (7) with $C = 0$ leads to five independent equations for the unknowns A, B, D, k and c provided $n = 2$ and $m = 1$. The solutions (5) are then recovered. A related method was used by Huang *et al* [11]. They assumed that

$$U(\xi) = \sum_{i=0}^{M} a_i T^i. \tag{13}$$

Substitution of (13) with $M = 2$ into (7) with $C \neq 0$ leads to five equations for the unknown constants a_0, a_1, a_2, k and c. The resulting solutions are (10). Malfliet [12] has proposed this method (with $C = 0$) as a systematic approach to finding solitary wave solutions to a wide variety of nonlinear wave equations. (Malfliet omits the factor $\frac{1}{2}$ in the definition of T in (4); it is included here so as to make notation consistent within this article.) This hyperbolic tangent method has been applied successfully to equations considerably more complicated than (2), see for example [13, 14] and the related paper [15].

We now turn our attention to the two-dimensional problem. We present two efficient methods of obtaining solutions to the 2D-KdVB equation (1) and comment on two other methods of solution that have been reported recently [1, 2].

We generalize the definition of ξ to $\xi = kx + ly - \omega t$. On taking $u(x, y, t) = U(\xi)$ in (1) we obtain

$$-cU + \tfrac{1}{2}U^2 + \mu k^2 U_{\xi\xi} - \nu k U_\xi = C \qquad \text{where} \qquad c = (\omega k - sl^2)/k^2 \tag{14}$$

and C is an integration constant. We note that (14) is identical in form to (7) so that efficient methods based on (7) that worked for (2) will also work for (1) provided that the different expression for c is taken into account; here we present details of the hyperbolic tangent method and the Ince transformation method as applied to the 2D-KdVB equation.

To apply the two-dimensional generalization of the hyperbolic tangent method we substitute (13) into (14) and take $C = 0$. The left-hand side of (14) becomes a power series in T. On balancing the highest order contributions from the linear terms with the highest order contributions from the nonlinear terms we find that $M = 2$. We equate coefficients of T^i ($i = 4, 3, 2, 1, 0$) to obtain the following five equations for the unknowns a_0, a_1, a_2, k and c.

$$\begin{aligned}
&\tfrac{1}{2}a_2^2 + \tfrac{3}{2}\mu k^2 a_2 = 0\\
&a_1 a_2 + \tfrac{1}{2}\mu k^2 a_1 + \nu k a_2 = 0\\
&-ca_2 + \tfrac{1}{2}a_1^2 + a_0 a_2 - 2\nu k^2 a_2 + \tfrac{1}{2}\nu k a_1 = 0\\
&-ca_1 + a_0 a_1 - \tfrac{1}{2}\mu k^2 a_1 - \nu k a_2 = 0\\
&-ca_0 + \tfrac{1}{2}a_0^2 + \tfrac{1}{2}\mu k^2 a_2 - \tfrac{1}{2}\nu k a_1 = 0.
\end{aligned}$$

From these equations we find

$$a_0 = \frac{3\nu^2}{25\mu} \pm \frac{6\nu^2}{25\mu} \qquad a_1 = -\frac{6\nu^2}{25\mu} \qquad a_2 = -\frac{3\nu^2}{25\mu} \qquad k = \frac{\nu}{5\mu} \qquad c = \pm\frac{6\nu^2}{25\mu}.$$

It follows that

$$u = \frac{3\nu^2}{25\mu}[S^2 - 2T \pm 2] \qquad \text{with} \qquad \xi = \frac{\nu}{5\mu}x + ly - \left(\pm\frac{6\nu^3}{125\mu^2} + \frac{5sl^2\mu}{\nu}\right)t \tag{15}$$

where l is arbitrary. The solutions (15) are the two-dimensional versions of the solutions (5). We may generate other solutions to (1) from (15) by using the fact that (1) is invariant under the transformation

$$u^* = u + \lambda \qquad x^* = x + \lambda t \qquad t^* = t \qquad y^* = y. \tag{16}$$

In particular, with λ given by (11), we obtain the solution corresponding to $C \neq 0$ in (15), namely

$$u = \frac{3\nu^2}{25\mu}[S^2 - 2T] + c \tag{17}$$

where

$$\xi = \frac{\nu}{5\mu}x + ly - \left(\frac{\nu c}{5\mu} + \frac{5sl^2\mu}{\nu}\right)t \qquad \text{and} \qquad c = \pm\left[\left(\frac{6\nu^2}{25\mu}\right)^2 - 2C\right]^{1/2}.$$

The solutions (17) may be written in the alternative form

$$u = -\frac{12\nu^2}{25\mu}\frac{1}{\{1+\exp[-(\xi-\xi_0)]\}^2} + c + \frac{6\nu^2}{25\mu} \tag{18}$$

with ξ and c as in (17).

To apply the Ince transformation method we substitute (8) into (14) which then reduces to

$$\mu k^2 r'' + \tfrac{1}{2}r^2 = 0 \tag{19}$$

provided that

$$k = \frac{\nu}{5\mu} \qquad \beta = c + \frac{6\nu^2}{25\mu} \qquad c = \pm\left[\left(\frac{6\nu^2}{25\mu}\right)^2 - 2C\right]^{1/2}. \tag{20}$$

Equation (19) may be integrated twice to give

$$r = -\frac{12\mu k^2}{(z+E)^2} \tag{21}$$

where, to ensure that the solution for r is bounded, the first constant of integration is taken to be zero and the second constant of integration E is taken to be strictly positive. Combining (8), (20) and (21) and setting $E = \exp(\xi_0)$ we obtain (18).

Li and Wang [1] transformed (1) using (3). The choice $p = -12\nu/5$ and $q = 12\mu$ leads to a long and complicated equation in which every term is of degree four in terms of F and its derivatives. On substituting $F = 1 + \exp[\xi - \xi_0]$ into this equation they found four equations that have to be satisfied by ω, k and l; *Mathematica* was used to find ω and k with l arbitrary. Eventually this method leads to a pair of solutions given by (12) and (13) in [1] and by (15) in this article. Ma [2] introduced the transformation

$$U = \frac{\alpha F_x^2}{F^2} + c + (c^2 + 2C)^{1/2}$$

into (14). The resulting equation for F is satisfied by $F = 1 + \exp[\xi - \xi_0]$ provided that

$$\alpha = -12\mu k^2 \qquad k = \frac{\nu}{5\mu} \qquad c = \pm\left[\left(\frac{6\nu^2}{25\mu}\right)^2 - 2C\right]^{1/2}.$$

The outcome is a pair of solutions given by (10) in [2] and by (18) in this article.

We have shown that, in view of (16), the solutions to the 2D-KdVB equation (1) obtained in [1] and [2] are equivalent. The method of solution used in [1] is tedious. The method of solution in [2] and the Ince transformation method described herein are more efficient but are dependent on the particular form of (14). The two-dimensional generalization of the hyperbolic tangent method described herein is not only efficient but also has the merit of being widely applicable.

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